

ON THE EXPONENTIAL STABILITY OF SWITCHING-DIFFUSION PROCESSES WITH JUMPS

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ABSTRACT. In this paper we focus on the pathwise stability of mild solutions for a class of stochastic partial differential equations which are driven by switching-diffusion processes with jumps. In comparison to the existing literature, we show that: (i) the criterion to guarantee pathwise stability does not rely on the moment stability of the system; (ii) the sample Lyapunov exponent obtained is generally smaller than that of the counterpart driven by a Wiener process; (iii) due to the Markovian switching the overall system can become pathwise exponentially stable although some subsystems are not stable.

1. INTRODUCTION

Stochastic partial differential equations (SPDEs) have been widely used to model phenomena arising in many branches of science such as ecology, economics, mechanics, biology and chemistry, e.g., Applebaum [3], Chow [9], Da Prato and Zabczyk [10], Liu [16], Peszat and Zabczyk [24], and Woyczyński [23]. Recently, hybrid systems, in which continuous dynamics are intertwined with discrete events, have also been used to model many such systems. One of the distinct features of hybrid systems is that the underlying dynamics are subject to changes with respect to certain configurations. For example, consider a one-dimensional rod of length π whose ends are maintained at 0° and whose sides are insulated. Assume that there is an exothermic reaction taking place inside the rod with heat being produced proportionally to the temperature. The temperature u in the rod may be modelled by (see, e.g., [11])

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + cu, & t > 0, \quad x \in (0, \pi), \\ u(t, 0) = u(t, \pi) = 0, \\ u(0, x) = u_0(x), \end{cases}$$

where $u = u(t, x)$ and c is a constant dependent on the rate of reaction. The system (1.1) will switch from one mode to another in a random way when it experiences abrupt changes in its structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, or abrupt environmental disturbances. A hybrid system driven by a continuous-time Markov chain can be applied to describe such a situation. The system (1.1) under regime

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switching could be described by the following stochastic model

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c(r(t))u, & t > 0, \quad x \in (0, \pi), \\ u(t, 0) = u(t, \pi) = 0, \\ u(0, x) = u_0(x), \quad r(0) = r_0, \end{cases}$$

where $r(t)$ is a right-continuous Markov chain with finite state space \mathbb{S} and $c : \mathbb{S} \rightarrow \mathbb{R}$. As a second example, Li et al. [14] discussed stochastic age-dependent population equation with Markovian switching,

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{\partial P}{\partial a} = -\mu(r(t), a)P + f(r(t), P) + g(r(t), P) \frac{dW(t)}{dt}, & (t, a) \in Q, \\ P(0, a) = P_0(a), \quad r(0) = r_0, & a \in [0, A], \\ P(t, 0) = \int_0^A \beta(t, a)P(t, a)da, & t \in [0, T], \end{cases}$$

where $T > 0, A > 0, Q := (0, T) \times (0, A), P = P(t, a)$ is the population density of age a at time t , $r(t)$ is a right-continuous Markov chain, $\mu(r(t), a)$ denotes the mortality rate of age a at time t , and $f(r(t), P)$ denotes the effects of the external environment on the population system. For more quantitative analysis of SPDEs with Markovian regime switching, we refer to Anabtawi and Ladde [1], Anabtawi and Sathananthan [2], Luo and Liu [17], and the references therein. For the finite-dimensional case, we refer to the monographs of Mao and Yuan [18] and Yin and Zhu [25].

Non-Gaussian random processes also play an important role in modelling stochastic dynamical systems, e.g., Applebaum [3], Øksendal and Sulem [21], and Peszat and Zabczyk [24]. Typical examples of non-Gaussian stochastic processes are Lévy processes and processes arising from Poisson random measures. The monograph [23] describes a number of phenomena from fluid mechanics, solid state physics, polymer chemistry, economic science, etc., which can be modelled using non-Gaussian Lévy processes.

Moreover, one of the most important and interesting problems in the analysis of SPDEs is their stability. Stability issues for SPDEs with Wiener noise are by now classical, see, e.g., [8, 9, 13, 16, 17], but comparable theories driven by jump noise are not yet fully developed. Recent years have witnessed a growing interest in this area: In [6] we investigated the asymptotic stability in distribution of mild solutions for delay equations using Lyapunov functions and the Yorsida approximation; by energy inequality approach, moment stability and sample path stability for variational solutions were discussed in [12], and almost sure exponential stability was studied in [17] provided that the mild solution is moment exponentially stable. For the finite-dimensional case, Applebaum and Siakalli [4] provide sufficient conditions under which the solutions to stochastic differential equations (SDEs) driven by Lévy noise are stable in probability, almost surely and moment exponentially stable. As a sequel, Applebaum and Siakalli [5] made some first steps in the stochastic stabilization problems where the noise source is a Lévy noise, i.e., a Brownian motion and an independent Poisson random measure.

Most of the previous literature does not discuss the sample Lyapunov exponents which are given explicitly by the parameters arising from the jump-diffusion coefficients or stationary probability distribution of Markovian chain. In this paper, motivated by the previous references, we shall study the pathwise stability of mild

solutions for a class of SPDEs of the form

$$(1.2) \quad dX(t) = [AX(t) + F(t, X(t), r(t))]dt + G(t, X(t), r(t))dW(t) + \int_{\mathbb{Z}} \Phi(t, X(t^-), r(t), u) \tilde{N}(dt, du)$$

on $t \geq 0$ with the initial data $X(0) = x_0 \in H$ and $r(0) = r_0 \in \mathbb{S}$, where $X(t^-) := \lim_{s \uparrow t} X(s)$. More detailed information on the parameters in Eq. (1.2) will be given in Section 2.

In comparison to the existing literature for the almost sure exponential stability of solutions to SPDEs, our main result (Theorem 3.1), has the following advantages: (i) the criterion established does not rely on the moment exponential stability of the system; (ii) the sample Lyapunov exponent is generally smaller than that of the counterpart driven by a Wiener process; (iii) due to the Markovian switching the overall system could become pathwise exponentially stable, although some subsystems are not stable.

Our approach is based on the Yosida approximation and a classical Lyapunov function argument. Since mild solutions do not necessarily have stochastic differentials, one cannot apply the Itô formula directly. To overcome this problem, we first apply the Itô formula to an approximating equation, and then investigate the stability properties of the mild solutions. This approach is dependent on a maximal inequality (Burkholder-Davis-Gundy inequality) for stochastic convolutions with jumps.

2. PRELIMINARIES

Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a Hilbert space and $W(t)$ a cylindrical Wiener process on H defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let \mathbb{Z} be a vector space endowed with a norm $\|\cdot\|$, $\mathcal{B}(\mathbb{Z})$ is the Borel σ -algebra on \mathbb{Z} , and $\lambda(dx)$ a σ -finite measure defined on $\mathcal{B}(\mathbb{Z})$. Let $p = (p(t)), t \in D_p$, be a stationary \mathcal{F}_t -Poisson point process on \mathbb{Z} with characteristic measure λ . Denote by $N(dt, du)$ the Poisson counting measure associated with p , i.e., $N(t, \mathbb{Y}) = \sum_{s \in D_p, s \leq t} I_{\mathbb{Y}}(p(s))$ for $\mathbb{Y} \in \mathcal{B}(\mathbb{Z})$. Let $\tilde{N}(dt, du) := N(dt, du) - dt\lambda(du)$ be the compensated Poisson measure. Let m be some positive integer and $\{r(t), t \in \mathbb{R}_+\}$ a right continuous irreducible Markov chain on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} := \{1, 2, \dots, m\}$, with generator $\Gamma = (\gamma_{ij})_{m \times m}$ given by

$$\mathbb{P}(r(t + \Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j , if $i \neq j$; while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. For a mapping $\zeta : \mathbb{S} \rightarrow \mathbb{R}$, we write $\zeta_i := \zeta(i)$. We further assume that the Wiener process $W(t)$, Poisson process $N(t, \cdot)$ and Markov chain $r(t)$ are independent. For $T > 0$ and $p > 0$, let $D([0, T]; H)$ be the family of all càdlàg paths from $[0, T]$ into H with the uniform norm and $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}; D([0, T]; H)) := \{X : \mathbb{E} \sup_{0 \leq t \leq T} \|X(t)\|_H^p < \infty\}$.

Remark 2.1. Since we have assume that the Markov chain $r(t)$ is irreducible, it has a unique stationary probability distribution $\pi := (\pi_1, \pi_2, \dots, \pi_m) \in \mathbb{R}^{1 \times m}$ which

can be determined by solving

$$\pi\Gamma = 0 \quad \text{subject to} \quad \sum_{j=1}^m \pi_j = 1 \text{ and } \pi_j > 0, \quad \forall j \in \mathbb{S}.$$

Throughout the paper we will assume:

(H1) $A : \mathcal{D}(A) \rightarrow H$ is the infinitesimal generator of a pseudo-contraction C_0 -semigroup $S(t)$ on H , that is,

$$(2.1) \quad \|S(t)\| \leq e^{\alpha t}, \quad \text{for some } \alpha \geq 0.$$

(H2) $F : [0, T] \times H \times \mathbb{S} \rightarrow H$, $G : [0, T] \times H \times \mathbb{S} \rightarrow \mathcal{L}_2(H, H)$, the family of Hilbert-Schmidt operators from H into itself, and $\Phi : [0, T] \times H \times \mathbb{Z} \times \mathbb{S} \rightarrow H$, satisfy Lipschitz and linear growth conditions, i.e., there exist positive constants L, \tilde{L} such that for arbitrary $x, y \in H, i \in \mathbb{S}$

$$\|F(t, x, i) - F(t, y, i)\|_H^2 + \|G(t, x, i) - G(t, y, i)\|_{HS}^2 \leq L\|x - y\|_H^2,$$

$$\|F(t, x, i)\|_H^2 + \|G(t, x, i)\|_{HS}^2 \leq \tilde{L}(1 + \|x\|_H^2),$$

$$\int_{\mathbb{Z}} \|\Phi(t, x, i, u) - \Phi(t, y, i, u)\|_H^2 \lambda(du) \leq L\|x - y\|_H^2,$$

$$\int_{\mathbb{Z}} \|\Phi(t, x, i, u)\|_H^2 \lambda(du) \leq \tilde{L}(1 + \|x\|_H^2).$$

Moreover, we will need some slightly stronger conditions on Φ :

(H3) For $p \geq 2$, there exists $\bar{L} > 0$ such that

$$(2.2) \quad \int_{\mathbb{Z}} \|\Phi(t, x, i, u) - \Phi(t, y, i, u)\|_H^p \lambda(du) \leq \bar{L}\|x - y\|_H^p, \quad x, y \in H, i \in \mathbb{S},$$

and $\Phi(t, 0, i, u) = 0$.

(H3') There exists an $L > 0$ such that for arbitrary $x, y \in H, i \in \mathbb{S}, u \in \mathbb{Z}$ and $p \geq 2$

$$\|\Phi(t, x, i, u) - \Phi(t, y, i, u)\|_H \leq L\|x - y\|_H|u| \text{ and } \int_{\mathbb{Z}} |u|^p \lambda(du) < \infty.$$

Clearly (H3') implies (H3) and $\|\Phi(t, x, i, u)\|_H \leq L\|x\|_H|u|$ whenever $\Phi(t, 0, i, u) = 0$.

In this paper we are mainly concerned with mild solutions to (1.2). For the notion of mild solutions, we can refer to, e.g., [19, Definition 2.1], for SPDEs driven by multiplicative Poisson noise, and, with an obvious extension, we can define such solutions for the case of switching-diffusion processes with jumps.

Lemma 2.2. Under conditions (H1) and (H2), Eq. (1.2) admits a unique mild solution $X(t, x_0, r_0), t \in [0, T]$, in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$.

Proof. Since the proof of Lemma 2.2 can be done by combining classical Banach fixed-point theorem argument [24, Theorem 9.29, p164] with stopping time techniques [18, Theorem 3.13, p89], we will only give a sketch of the argument. Recall that almost every sample path of $r(\cdot)$ is a right-continuous step function with a finite number of sample jumps on $[0, T]$. So for almost every $\omega \in \Omega$ there is a finite $k = k(\omega)$ such that $0 = \tau_0 < \tau_1 < \dots < \tau_k \geq T$ and

$$r(t) = r(\tau_k) \text{ on } \tau_k \leq t < \tau_{k+1} \text{ for } \forall k \geq 0.$$

We first consider Eq. (1.2) on $t \in [0, \tau_1]$ which becomes

$$(2.3) \quad \begin{aligned} dX(t) &= [AX(t) + F(t, X(t), r_0)]dt + G(t, X(t), r_0)dW(t) \\ &\quad + \int_{\mathbb{Z}} \Phi(t, X(t^-), r_0, u)\tilde{N}(dt, du) \end{aligned}$$

with the initial data $X(0) = x_0 \in H$ and $r(0) = r_0 \in \mathbb{S}$. By [24, Theorem 9.29, p164] Eq. (2.3) admits a unique mild solution $X(t, x_0, r_0)$ for $t \in [0, \tau_1]$ which belongs to $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$. Next consider Eq. (3.1) on $t \in [\tau_1, \tau_2]$ which becomes

$$(2.4) \quad \begin{aligned} dX(t) &= [AX(t) + F(t, X(t), r(\tau_1))]dt + G(t, X(t), r(\tau_1))dW(t) \\ &\quad + \int_{\mathbb{Z}} \Phi(t, X(t^-), r(\tau_1), u)\tilde{N}(dt, du) \end{aligned}$$

with the initial data $X(\tau_1, x_0, r_0)$ and $r(\tau_1)$. Again by [24, Theorem 9.29, p164] Eq. (2.4) admits a unique mild solution $X(t, X(\tau_1, x_0, r_0), r(\tau_1))$ for $t \in [\tau_1, \tau_2]$ which belongs to $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$. The proof can then be completed by repeating this procedure. \square

Let $U : \mathbb{R}_+ \times H \times \mathbb{S} \rightarrow \mathbb{R}_+$ be a $C^{1,2}$ -function. For $t \geq 0, x \in \mathcal{D}(A)$ and $i \in \mathbb{S}$, define an operator

$$\begin{aligned} \mathcal{L}U(t, x, i) &:= U_t(t, x, i) + \langle Ax + F(t, x, i), U_x(t, x, i) \rangle_H \\ &\quad + \sum_{j=1}^m \gamma_{ij} U(t, x, j) + \frac{1}{2} \text{trace}(U_{xx}(t, x, i) G(t, x, i) G^*(t, x, i)) \\ &\quad + \int_{\mathbb{Z}} [U(t, x + \Phi(t, x, i, u), i) - U(t, x, i) - \langle U_x(t, x, i), \Phi(t, x, i, u) \rangle_H] \lambda(du). \end{aligned}$$

Similarly to [18, Theorem 1.45, p48] and [24, Theorem D.2, p392], for a strong solution $X(t)$ of Eq. (1.2), we have the following Itô formula:

$$(2.5) \quad \begin{aligned} &U(t, X(t), r(t)) \\ &= U(0, x_0, r_0) + \int_0^t \mathcal{L}U(s, X(s), r(s))ds \\ &\quad + \int_0^t \langle U_x(s, X(s), r(s)), G(s, X(s), r(s))dW(s) \rangle_H \\ &\quad + \int_0^t \int_{\mathbb{Z}} [U(s, X(s^-) + \Phi(s, X(s^-), r(s), u), r(s)) - U(s, X(s^-), r(s))] \tilde{N}(ds, du) \\ &\quad + \int_0^t \int_{\mathbb{R}} [U(s, X(s^-), r_0 + h(r(s), \ell)) - U(s, X(s^-), r(s))] \mu(ds, d\ell), \end{aligned}$$

where $\mu(ds, d\ell)$ is a Poisson random measure with intensity $ds \times m(d\ell)$, in which m is the Lebesgue measure on \mathbb{R} . For more details on the function h and the martingale measure $\mu(ds, d\ell)$, see, e.g., [18, p46-48].

Since mild solutions do not necessarily have stochastic differentials, one cannot apply the Itô formula directly. Instead, we introduce the Yosida-approximation

system

$$(2.6) \quad \begin{aligned} dX^l(t) = & [AX^l(t) + R(l)F(t, X^l(t), r(t))]dt + R(l)G(t, X^l(t), r(t))dW(t) \\ & + \int_{\mathbb{Z}} R(l)\Phi(t, X^l(t^-), r(t), u)\tilde{N}(dt, du) \end{aligned}$$

with the initial data $X^l(0) = R(l)x_0$ and $r(0) = r_0$. Here $l \in \rho(A)$, the resolvent set of A , and $R(l) := lR(l, A) := l(lId - A)^{-1}$, where Id is the identity operator from H into itself.

Theorem 2.3. Under conditions (H1) – (H3), Eq. (2.6) has a unique strong solution $X^l(t)$ in $L^p, p \geq 2$. Moreover, $X^l(t)$ converges to the mild solution $X(t)$ of Eq. (1.2) in L^p as $l \rightarrow \infty$, i.e.,

$$\lim_{l \rightarrow \infty} \mathbb{E} \left(\sup_{t \in [0, T]} \|X^l(t) - X(t)\|_H^p \right) = 0.$$

In particular, there exists a subsequence, still denoted by $X^l(t)$, such that $X^l(t) \rightarrow X(t)$ almost surely as $l \rightarrow \infty$ uniformly in $[0, T]$.

Proof. This can be proven by following the argument in [17, Proposition 2.4] and using the result of Lemma 2.4 below. \square

Lemma 2.4. ([7, Theorem 4.4] or [19, Proposition 3.3]) Assume that $\Phi : \Omega \times \mathbb{R}_+ \times \mathbb{Z} \rightarrow H$, is a progressively measurable process, and for $p \geq 2$

$$(2.7) \quad \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|\Phi(s, u)\|_H^p \lambda(du) ds < \infty.$$

If $S(t), t \in [0, T]$, is a pseudo-contraction C_0 -semigroup such that (2.1), then

$$(2.8) \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \int_{\mathbb{Z}} S(t-s)\Phi(s, u)\tilde{N}(ds, du) \right\|_H^p \right] \leq C_p \mathbb{E} \int_0^T \int_{\mathbb{Z}} \|\Phi(s, u)\|_H^p \lambda(du) ds,$$

where C_p is a positive constant dependent on p, α, T .

3. A CRITERION FOR SAMPLE PATH STABILITY

In this section we give a criterion for sample path stability of mild solutions to Eq. (1.2).

Theorem 3.1. Let (H1), (H2) and (H3') hold. Assume that the solution of Eq. (1.2) is such that $X(t) \neq 0$ a.s. for all $t \geq 0$ and $i \in \mathbb{S}$ provided $x_0 \neq 0$ a.s. For $U \in C^{1,2}(\mathbb{R}_+ \times H \times \mathbb{S}; \mathbb{R}_+)$, assume further that there exist constants $c_2 > c_1 > 0, c_3 > 0, p > 0, \alpha_i, \rho_i \in \mathbb{R}, \beta_i, \delta_i \geq 0$ such that for $(t, x, i) \in \mathbb{R}_+ \times \mathcal{D}(A) \times \mathbb{S}$

$$(i) \quad c_1 \|x\|_H^p \leq U(t, x, i) \leq c_2 \|x\|_H^p, \quad \|U_x(t, x, i)\|_H \|x\|_H + \|U_{xx}(t, x, i)\| \|x\|_H^2 \leq c_3 \|x\|_H^p;$$

$$(ii) \quad \mathcal{L}U(t, x, i) \leq \alpha_i U(t, x, i);$$

$$(iii) \quad \Theta U(t, x, i) := \|G^*(t, x, i)U_x(t, x, i)\|_H^2 \geq \beta_i U^2(t, x, i);$$

(iv) For $\Psi(t, x, i, j) := U(t, x, j)/U(t, x, i)$,

$$\sum_{j=1}^m \gamma_{ij} (\ln \Psi(t, x, i, j) - \Psi(t, x, i, j)) \leq \rho_i;$$

(v) For $\Lambda(t, x, i, u) := U(t, x + \Phi(t, x, i, u), i)/U(t, x, i)$,

$$\int_{\mathbb{Z}} [\ln \Lambda(t, x, i, u) - \Lambda(t, x, i, u) + 1] \lambda(du) := J(t, x, i) \leq -\delta_i,$$

and for some $\epsilon \in (0, \frac{1}{2}]$

$$\zeta := \limsup_{t \rightarrow \infty} \frac{1}{t} \Xi(t) < \infty,$$

where

$$\Xi(t) := \int_0^t \int_{\mathbb{Z}} [(\ln \Lambda(s, x, i, u))^2 + \Lambda^\epsilon(s, x, i, u)] \lambda(du) ds;$$

(vi) For $\Upsilon(t, x, r_0, i, \ell) := U(t, x, r_0 + h(i, \ell))/U(t, x, i)$ and some $\tilde{\epsilon} \in (0, \frac{1}{2}]$

$$\eta := \limsup_{t \rightarrow \infty} \frac{1}{t} \Pi(t) < \infty,$$

where

$$\Pi(t) := \int_0^t \int_{\mathbb{Z}} [(\ln \Upsilon(s, x, r_0, i, \ell))^2 + \Upsilon^{\tilde{\epsilon}}(s, x, r_0, i, \ell)] m(d\ell) ds.$$

Then the mild solution of Eq. (1.2) has the property

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) \leq -\frac{1}{p} \sum_{i=1}^m \pi_i \left(\frac{1}{2} \beta_i + \delta_i - \alpha_i - \rho_i \right), \quad a.s.$$

In particular, the mild solution of Eq. (1.2) is almost surely exponentially stable provided that

$$\sum_{i=1}^m \pi_i \left(\frac{1}{2} \beta_i + \delta_i - \alpha_i - \rho_i \right) > 0.$$

Remark 3.2. By the fundamental inequality

$$\ln(1+x) \leq x \quad \text{for } x \geq 0,$$

it is easy to observe that the first assumption in (v) is reasonable. On the other hand, for $x \in H$ and $i, j \in \mathbb{S}$, (vi) is also true provided that there exist constants $c_4, c_5 > 0$ such that

$$c_4 \leq U(t, x, j)/U(t, x, i) \leq c_5.$$

There are many functions possessing this property, e.g., for $x \in H, i \in \mathbb{S}, U(t, x, i) = \sigma_i \|x\|_H^2$ with $\sigma_i > 0$. For this Lyapunov function, we have $\Psi(t, x, i, j) = \sigma_j/\sigma_i$ and then (iv) also holds.

Proof of Theorem 3.1. As we mentioned previously, one cannot apply the Itô formula to the mild solutions directly. we first apply the Itô formula to the approximation equation (2.6), then use Theorem 2.3 to investigate the stability of the mild solutions. More precisely, applying the Itô formula (2.5) to $\ln U(t, x, i)$ with

respect to $X^l(t), t \geq 0$, where $X^l(t)$ denotes the strong solution of Eq. (2.6), we obtain

$$\begin{aligned}
\ln U(t, X^l(t), r(t)) &= \ln U(0, R(l)x_0, r_0) + \int_0^t \frac{\mathcal{L}U(s, X^l(s), r(s))}{U(s, X^l(s), r(s))} ds \\
&+ \int_0^t \sum_{j=1}^m \gamma_{r(s)j} [\ln \Psi(s, X^l(s), i, j) - \Psi(s, X^l(s), i, j)] ds \\
&+ \int_0^t \int_{\mathbb{Z}} \ln \left(\frac{U(s, X^l(s^-) + R(l)\Phi(s, X^l(s^-), r(s), u), r(s))}{U(s, X^l(s^-), r(s))} \right) \tilde{N}(ds, du) \\
&+ \int_0^t \int_{\mathbb{R}} \ln \Upsilon(t, X^l(s^-), r_0, r(s), \ell) \mu(ds, d\ell) \\
&+ J_1(t, l) + J_2(t, l) + J_3(t, l),
\end{aligned}$$

where

$$\begin{aligned}
J_1(t, l) &:= \frac{1}{2} \int_0^t \text{trace} \left[\left(\frac{U_{xx}(s, X^l(s), r(s))}{U(s, X^l(s), r(s))} - \frac{U_x(s, X^l(s), r(s)) \otimes U_x(s, X^l(s), r(s))}{U^2(s, X^l(s), r(s))} \right) \right. \\
&\quad \left. R(l)G(s, X^l(s), r(s))(R(l)G(s, X^l(s), r(s)))^* \right] ds \\
&\quad - \frac{1}{2} \int_0^t \frac{\text{trace} [U_{xx}(s, X^l(s), r(s))G(s, X^l(s), r(s))(G(s, X^l(s), r(s)))^*]}{U(s, X^l(s), r(s))} ds \\
J_2(t, l) &:= \int_0^t \frac{\langle U_x(s, X^l(s), r(s)), (R(l) - I)F(s, X^l(s), r(s)) \rangle_H}{U(s, X^l(s), r(s))} ds, \\
J_3(t, l) &:= \int_0^t \int_{\mathbb{Z}} \left[\ln \left(\frac{U(s, X^l(s) + R(l)\Phi(s, X^l(s), r(s), u), r(s))}{U(s, X^l(s), r(s))} \right) \right. \\
&\quad - \frac{\langle U_x(s, X^l(s), r(s)), (R(l) - I)\Phi(s, X^l(s), r(s), u), r(s) \rangle}{U(s, X^l(s), r(s))} \\
&\quad \left. - \frac{U(s, X^l(s) + \Phi(s, X^l(s), r(s), u), r(s))}{U(s, X^l(s), r(s))} + 1 \right] \lambda(du) ds.
\end{aligned}$$

By Theorem 2.3, (H3'), (i) and the Dominated Convergence Theorem, we have almost surely

$$\begin{aligned}
\lim_{l \rightarrow \infty} J_1(t, l) &= -\frac{1}{2} \int_0^t \frac{\Theta U(s, X(s), r(s))}{U(s, X(s), r(s))} ds, \quad \lim_{l \rightarrow \infty} J_2(t, l) = 0, \\
\lim_{l \rightarrow \infty} J_3(t, l) &= \int_0^t J(s, X(s), r(s)) ds.
\end{aligned}$$

If we let $l \rightarrow \infty$, then,

$$\begin{aligned}
\ln U(t, X(t), r(t)) &= \ln U(0, x_0, r_0) + \int_0^t J(s, X(s), r(s)) ds \\
&+ \int_0^t \frac{\mathcal{L}U(s, X(s), r(s))}{U(s, X(s), r(s))} ds - \frac{1}{2} \int_0^t \frac{\Theta U(s, X(s), r(s))}{U^2(s, X(s), r(s))} ds \\
&+ \int_0^t \frac{\langle U_x(s, X(s), r(s)), g(s, X(s), r(s)) dW(s) \rangle_H}{U(s, X(s), r(s))} \\
&+ \int_0^t \sum_{j=1}^m \gamma_{r(s)j} [\ln \Psi(s, X(s), i, j) - \Psi(s, X(s), i, j)] ds \\
&+ \int_0^t \int_{\mathbb{Z}} \ln \Lambda(s, X(s^-), r(s), u) \tilde{N}(ds, du) \\
&+ \int_0^t \int_{\mathbb{R}} \ln \Upsilon(s, X(s^-), r_0, r(s), \ell) \mu(ds, d\ell).
\end{aligned}$$

By the exponential martingale inequality with jumps [3, Theorem 5.2.9, p291], for any $T, \theta, \nu > 0$

$$\begin{aligned}
&\mathbb{P}\left\{\omega : \sup_{0 \leq t \leq T} \left[\int_0^t \frac{\langle U_x(s, X(s), r(s)), g(s, X(s), r(s)) dW(s) \rangle_H}{U(s, X(s), r(s))} - \frac{\theta^2}{2} \int_0^t \frac{\Theta U(s, X(s), r(s))}{U^2(s, X(s), r(s))} ds \right. \right. \\
&\quad + \int_0^t \int_{\mathbb{Z}} \ln \Lambda(s, X(s^-), r(s), u) \tilde{N}(ds, du) \\
&\quad \left. \left. - \frac{1}{\theta} \int_0^t \int_{\mathbb{Z}} [\Lambda^\theta(s, X(s), r(s), u) - 1 - \theta \ln \Lambda(s, X(s), r(s), u)] \lambda(du) ds \right] > \nu \right\} \\
&\leq e^{-\theta\nu},
\end{aligned}$$

and, if we denote $\tilde{\Upsilon}(t, \ell) := \Upsilon(t, x, r_0, i, \ell)$, then

$$\begin{aligned}
&\mathbb{P}\left\{\omega : \sup_{0 \leq t \leq T} \left[\int_0^t \int_{\mathbb{R}} \ln \tilde{\Upsilon}(s, \ell) \mu(ds, d\ell) \right. \right. \\
&\quad \left. \left. - \frac{1}{\theta} \int_0^t \int_{\mathbb{R}} [\tilde{\Upsilon}^\theta(s, \ell) - 1 - \theta \ln \tilde{\Upsilon}(s, \ell)] m(d\ell) ds \right] > \nu \right\} \leq e^{-\theta\nu}.
\end{aligned}$$

Taking $T = n$, $\nu = 2\theta^{-1} \ln n$, $n = 1, 2, \dots$, for $\theta \in (0, \frac{\epsilon \wedge \tilde{\epsilon}}{2}]$ and applying the Borel-Cantelli Lemma, we see that there exists an $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $n_0 = n_0(\omega) > 0$ if $n \geq n_0$

$$\begin{aligned}
&\int_0^t \frac{\langle U_x(s, X(s), r(s)), g(s, X(s), r(s)) dW(s) \rangle_H}{U(s, X(s), r(s))} + \int_0^t \int_{\mathbb{Z}} \ln \Lambda(s, X(s^-), r(s), u) \tilde{N}(ds, du) \\
&\leq 2\theta^{-1} \ln n + \frac{\theta^2}{2} \int_0^t \frac{\Theta U(s, X(s), r(s))}{U^2(s, X(s), r(s))} ds \\
&\quad + \frac{1}{\theta} \int_0^t \int_{\mathbb{Z}} [\Lambda^\theta(s, X(s), r(s), u) - 1 - \theta \ln \Lambda(s, X(s), r(s), u)] \lambda(du) ds
\end{aligned}$$

and

$$\int_0^t \int_{\mathbb{R}} \ln \tilde{\Upsilon}(s, \ell) \mu(ds, d\ell) \leq 2\theta^{-1} \ln n + \frac{1}{\theta} \int_0^t \int_{\mathbb{R}} [\tilde{\Upsilon}^\theta(s, \ell) - 1 - \theta \ln \tilde{\Upsilon}(s, \ell)] m(d\ell) ds$$

for $0 \leq t \leq n$. Hence for $\omega \in \Omega_0$, $0 \leq t \leq n$ and $n \geq n_0$

$$\begin{aligned}
\ln U(t, X(t), r(t)) &\leq \ln U(0, x_0, r_0) + 4\theta^{-1} \ln n + \int_0^t J(s, X(s), r(s)) ds \\
&\quad + \int_0^t \frac{\mathcal{L}U(s, X(s), r(s))}{U(s, X(s), r(s))} ds - \frac{1-\theta}{2} \int_0^t \frac{\Theta U(s, X(s), r(s))}{U^2(s, X(s), r(s))} ds \\
&\quad + \int_0^t \sum_{j=1}^m \gamma_{r(s)j} [\ln \Psi(s, X(s), i, j) - \Psi(s, X(s), i, j)] ds \\
&\quad + \frac{1}{\theta} \int_0^t \int_{\mathbb{Z}} [\Lambda^\theta(s, X(s), r(s), u) - 1 - \theta \ln \Lambda(s, X(s), r(s), u)] \lambda(du) ds \\
&\quad + \frac{1}{\theta} \int_0^t \int_{\mathbb{R}} [\tilde{\Upsilon}^\theta(s, \ell) - 1 - \theta \ln \tilde{\Upsilon}(s, \ell)] m(d\ell) ds.
\end{aligned}$$

This, together with (ii)-(v), yields that for $\omega \in \Omega_0$, $0 \leq t \leq n$ and $n \geq n_0$

$$\begin{aligned}
\ln U(t, X(t), r(t)) &\leq \ln U(0, x_0, r_0) + 4\theta^{-1} \ln n \\
&\quad + \int_0^t [\alpha(r(s)) - \frac{1-\theta}{2} \beta(r(s)) - \delta(r(s)) + \rho(r(s))] ds \\
&\quad + \frac{1}{\theta} \int_0^t \int_{\mathbb{Z}} \Gamma(s, X(s), y, r(s), \theta) \lambda(dy) ds \\
&\quad + \frac{1}{\theta} \int_0^t \int_{\mathbb{R}} \tilde{\Theta}(s, \ell, \theta) m(d\ell) ds \\
&:= I_1(t) + I_2(t) + I_3(t) + I_4(t),
\end{aligned} \tag{3.1}$$

where

$$\Gamma(s, x, i, u, \theta) := \Lambda^\theta(s, x, i, u) - 1 - \theta \ln \Lambda(s, x, i, u)$$

and

$$\tilde{\Theta}(s, \ell, \theta) := \tilde{\Upsilon}^\theta(s, \ell) - 1 - \theta \ln \tilde{\Upsilon}(s, \ell).$$

Using Taylor's series expansion, for $\theta \in (0, \frac{\epsilon \wedge \tilde{\epsilon}}{2}]$ we see that

$$\Lambda^\theta(s, x, i, u) = 1 + \theta \ln \Lambda(s, x, i, u) + \frac{\theta^2}{2} (\ln \Lambda(s, x, i, u))^2 \Lambda^\xi(s, x, i, u),$$

where ξ lies between 0 and θ . Hence

$$\begin{aligned}
I_3(t) &= \frac{\theta}{2} \int_0^t \int_{\mathbb{Z}} (\ln \Lambda(s, X(s), r(s), u))^2 \Lambda^\xi(s, X(s), r(s), u) \lambda(du) ds \\
&= \frac{\theta}{2} \int_0^t \int_{0 < \Lambda \leq 1} (\ln \Lambda(s, X(s), r(s), u))^2 \Lambda^\xi(s, X(s), r(s), u) \lambda(du) ds \\
&\quad + \frac{\theta}{2} \int_0^t \int_{\Lambda > 1} (\ln \Lambda(s, X(s), r(s), u))^2 \Lambda^\xi(s, X(s), r(s), u) \lambda(du) ds.
\end{aligned}$$

Noting that, for $0 \leq \xi \leq \frac{\epsilon}{2}$, $\Lambda^\xi \leq 1$ if $0 < \Lambda \leq 1$, $\Lambda^\xi \leq \Lambda^{\frac{\epsilon}{2}}$ if $\Lambda \geq 1$, and recalling the inequality

$$\ln x \leq \frac{4}{\epsilon} (x^{\frac{\epsilon}{4}} - 1) \quad \text{for } x \geq 1,$$

we obtain

$$I_3(t) \leq \frac{\theta}{2} \int_0^t \int_{\mathbb{Z}} \left[(\ln \Lambda(s, X(s), r(s), u))^2 + \frac{16}{\epsilon^2} \Lambda^\epsilon(s, X(s), r(s), u) \right] \lambda(du) ds.$$

Similarly,

$$I_4(t) \leq \frac{\theta}{2} \int_0^t \int_{\mathbb{R}} \left[(\ln \Upsilon(s, \ell))^2 + \frac{16}{\tilde{\epsilon}^2} \Upsilon^{\tilde{\epsilon}}(s, \ell) \right] m(d\ell) ds.$$

Hence, by (i) for $\omega \in \Omega_0$, $n-1 \leq t \leq n$ and $n \geq n_0 + 1$

$$\begin{aligned} \frac{1}{t} \ln(\|X(t)\|_H) &\leq -\frac{\ln \rho}{pt} + \frac{1}{pt} \left[\ln U(0, x_0, r_0) + 4\theta^{-1} \ln n + \frac{8\theta}{(\epsilon \wedge \tilde{\epsilon})^2} (\Xi(t) + \Pi(t)) \right. \\ &\quad \left. + \int_0^t [\alpha(r(s)) - \frac{1-\alpha}{2} \beta(r(s)) - \delta(r(s)) + \rho(r(s))] ds \right]. \end{aligned}$$

Taking into account the ergodic property of Markovian chains, e.g., [20, Theorem 3.8.1, p126], and combining (v) with (vi), we have almost surely

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) \leq \frac{1}{p} \left[\frac{8\theta}{(\epsilon \wedge \tilde{\epsilon})^2} (\zeta + \eta) + \sum_{i=1}^m \pi_i \left(\alpha_i - \frac{1-\theta}{2} \beta_i - \delta_i + \rho_i \right) \right],$$

and the conclusion follows by the arbitrariness of θ .

Now we give an example to demonstrate Theorem 3.1.

Example 3.3. Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with the generator $\Gamma = (q_{ij})_{2 \times 2}$:

$$-q_{11} = q_{12} = 1, \quad -q_{22} = q_{21} = q > 0.$$

The unique stationary probability distribution of the Markov chain $r(t)$ is

$$\pi = (\pi_1, \pi_2) = (q/(1+q), 1/(1+q)).$$

Let $f, g : \mathbb{R} \times \mathbb{S} \rightarrow \mathbb{R}$ be Lipschitz continuous in the first argument and satisfy linear growth conditions. Assume that there exist constants $b_i \in \mathbb{R}, d_i > 0, i = 1, 2$, such that for $x \in \mathbb{R}$

$$(3.2) \quad 2xf(x, i) + g^2(x, i) \leq b_i x^2,$$

and

$$(3.3) \quad xg(x, i) \geq d_i^{\frac{1}{2}} x^2.$$

For $i = 1, 2$ let

$$\begin{aligned} \delta_i &:= \int_0^\infty [\gamma_i^2(y) + 2\gamma_i(y) - 2\ln(1 + \gamma_i(y))] \lambda(dy) > 0, \\ m_i &:= \int_0^\infty [2\gamma_i(y) - 2\ln(1 + \gamma_i(y))] \lambda(dy). \end{aligned}$$

Assume further that

$$(3.4) \quad \int_0^\infty [(\ln(1 + \gamma_i(y)))^2 + \gamma_i^2(y)] \lambda(dy) < \infty, \quad i = 1, 2.$$

Consider the following equation

$$\begin{aligned} dX(t) &= [AX(t) + f(X(t), r(t))] dt + g(X(t), r(t)) dW(t) \\ (3.5) \quad &+ \int_0^\infty \gamma(r(t), y) X(t^-) \tilde{N}(dy, dt), \quad t > 0, x \in (0, \pi); \\ X(0, x) &= u_0(x), \quad x \in (0, \pi); \quad X(t, 0) = X(t, \pi) = 0, \quad t \geq 0. \end{aligned}$$

In this example, set $H := L^2([0, \pi])$, $A = \frac{\partial}{\partial x}(a(x)\frac{\partial}{\partial x})$ with domain $\mathcal{D}(A)$ satisfying the boundary conditions above. We let $a(x)$ be a measurable function defined on $[0, \pi]$ such that

$$(3.6) \quad 0 < \nu \leq a(x),$$

for some positive constant ν .

Let $U(t, u, i) := \lambda_i \|u\|_H^2$, $u \in H$, $i = 1, 2$, where $\lambda_1 = 1$ and λ_2 is a positive constant which will be determined later. Note from (3.2), (3.6) and Poincaré's inequality that for $u \in \mathcal{D}(A)$

$$\begin{aligned} \mathcal{L}U(t, u, 1) &= 2\langle Au + f(u, 1), u \rangle_H + \|g(u, 1)\|_H^2 \\ &\quad + \int_0^\infty \gamma_1^2(y)\lambda(dy)\|u\|_H^2 + q_{11}\lambda_1\|u\|_H^2 + q_{12}\lambda_2\|u\|_H^2 \\ &\leq \left[-2\nu + b_1 + \int_0^\infty \gamma_1^2(y)\lambda(dy) + \lambda_2 - 1 \right] U(t, u, 1) \\ &:= \alpha_1 U(t, u, 1), \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{L}U(t, u, 2) &\leq \left[-2\nu + b_2 + \int_0^\infty \gamma_2^2(y)\lambda(dy) + q\left(\frac{1}{\lambda_2} - 1\right) \right] U(t, u, 2) \\ &:= \alpha_2 U(t, u, 2). \end{aligned}$$

By the definition of U , it is easy to see that

$$\rho_1 = 1 - \lambda_2 + \ln \lambda_2 \text{ and } \rho_2 = q\left(1 - \frac{1}{\lambda_2} - \ln \lambda_2\right).$$

From (3.3), it follows that

$$\Theta U(t, u, i) \geq d_i \|u\|_H^4 = \frac{d_i}{\lambda_i^2} U^2(t, u, i) := \beta_i U^2(t, u, i).$$

Moreover, (v) follows from (3.4) and (iv), and (vi) holds due to the definition of U . Thus, by Theorem 3.1 we arrive at

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) \leq -\frac{\vartheta}{2(1+q)}, \quad \text{a.s.},$$

where

$$\vartheta := q\left(\frac{d_1}{2} + m_1 + 2\nu - b_1\right) + \frac{d_2}{2\lambda_2^2} + m_2 + 2\nu - b_2.$$

In particular, let

$$\frac{d_1}{2} + m_1 + 2\nu - b_1 < 0,$$

and choose $\lambda_2 > 0$ such that

$$\frac{d_2}{2\lambda_2^2} + m_2 + 2\nu - b_2 > 0.$$

Then Eq. (3.5) is almost surely exponentially stable whenever

$$0 < q < -\left(\frac{d_2}{2\lambda_2^2} + m_2 + 2\nu - b_2\right) / \left(\frac{d_1}{2} + m_1 + 2\nu - b_1\right).$$

4. LINEAR SWITCHING-DIFFUSION SPDES WITH JUMPS

In this section to demonstrate that the results obtained in Theorem 3.1 are sharp, we will discuss a class of linear switching-diffusion SPDEs with jumps.

For a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ with C^∞ boundary $\partial\mathcal{O}$, let $H := L^2(\mathcal{O})$ denote the family of all real-valued square integrable functions, equipped with the usual inner product $\langle f, g \rangle_H := \int_{\mathcal{O}} f(x)g(x)dx$, $f, g \in H$ and norm $\|f\|_H := (\int_{\mathcal{O}} f^2(x)dx)^{\frac{1}{2}}$, $f \in H$. Let $A := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the classical Laplace operator defined on the Sobolev space $\mathcal{D}(A) := H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$, where $H^m(\mathcal{O})$, $m = 1, 2$, consist of functions of H whose derivatives $D^\alpha u$, in the sense of distributions, of order $|\alpha| \leq m$ are in H . It is well known that there exists an orthonormal basis of H , $\{e_n\}_{n \geq 1}$, $n = 1, 2, \dots$, satisfying

$$(4.1) \quad e_n \in \mathcal{D}(A), \quad -Ae_n = \lambda_n e_n.$$

Thus, for any $f \in H$, we can write

$$f = \sum_{n=1}^{\infty} f_n e_n, \quad \text{where } f_n = \langle f, e_n \rangle.$$

Consider the following SPDE driven by a switching-diffusion process with jumps

$$(4.2) \quad \begin{cases} dX(t) = (AX(t) + \bar{\alpha}(r(t))X(t))dt + \bar{\beta}(r(t))X(t)dW(t) \\ \quad + \int_0^\infty \bar{\gamma}(r(t), y)X(t^-)\tilde{N}(dt, dy), x \in \mathcal{O}, t > 0, \\ X(t, x) = 0, x \in \partial\mathcal{O}, t > 0, \\ X(0, x) = u^0(x), x \in \mathcal{O} \text{ and } r(0) = r_0. \end{cases}$$

Here $\bar{\alpha}, \bar{\beta} : \mathbb{S} \rightarrow \mathbb{R}$, $\bar{\gamma} : \mathbb{S} \times [0, \infty) \rightarrow \mathbb{R}$, W is a real-valued Wiener process on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.

Applying Theorem 3.1 we can deduce the following conclusion.

Theorem 4.1. Assume that for $i \in \mathbb{S}, y \in (0, \infty)$

$$(4.3) \quad \bar{\gamma}_i(y) := \bar{\gamma}(i, y) > -1, \quad \max_{1 \leq i \leq m} \int_0^\infty \bar{\gamma}_i^2(y) \lambda(dy) < \infty,$$

and

$$(4.4) \quad \max_{1 \leq i \leq m} \int_0^\infty (\ln(1 + \bar{\gamma}_i(y)))^2 \lambda(dy) < \infty.$$

Then Eq. (4.2) has the property

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) \leq -\lambda_1 + \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right) \quad a.s.$$

In particular, the solution of Eq. (4.2) is almost surely exponentially stable if

$$\lambda_1 > \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right).$$

Proof. Under (4.3), by Lemma 2.2 Eq. (4.2) admits a unique global mild solution. Let $U(u) := \|u\|_H^2$, $u \in H$, and set for $t \geq 0, u \in H, i \in \mathbb{S}$ and $y \in \mathbb{Z}$

$$A(t, u, i) := Au + \bar{\alpha}_i u, \quad g(t, u, i) := \bar{\beta}_i u \quad \text{and} \quad \Phi(t, u, i, y) := \bar{\gamma}_i(y)u.$$

It is easy to compute α_i, β_i and δ_i in Theorem 3.1 as follows

$$\alpha_i = -2\lambda_1 + 2\bar{\alpha}_i + \bar{\beta}_i^2 + \int_0^\infty \bar{\gamma}_i^2(y)\lambda(dy), \quad \beta_i = 4\bar{\beta}_i^2,$$

and

$$\delta_i = \int_0^\infty [2(\bar{\gamma}_i(y) - \ln(1 + \bar{\gamma}_i(y))) + \bar{\gamma}_i^2(y)]\lambda(dy).$$

Moreover, noting that

$$\Lambda(t, u, i, y) = (1 + \bar{\gamma}_i(y))^2 \text{ and } \Upsilon(t, u, r_0, i, \ell) = 1,$$

together with (4.3) and (4.4), we can deduce that (iv) and (vi) hold. Then the desired assertion follows by Theorem 3.1. \square

We now discuss the sample path stability of the solution to Eq. (4.2) using its explicit mild solution, which will be given in Lemma 4.2 below. In the sequel, when u^0 is deterministic and $u^0 \neq 0$, we set $n_0 := \inf\{n : u_n^0 \neq 0\}$, where $u_n^0 := \langle u^0, e_n \rangle$ for $n \geq 1$.

Lemma 4.2. Under (4.3), the unique global mild solution of Eq. (4.2) has the explicit form

(4.5)

$$\begin{aligned} X(t, x) = v(t, x) \exp \left\{ -\frac{1}{2} \int_0^t \bar{\beta}^2(r(s))ds + \int_0^t \int_0^\infty [\ln(1 + \bar{\gamma}(r(s), y)) - \bar{\gamma}(r(s), y)]\lambda(dy)ds \right. \\ \left. + \int_0^t \bar{\beta}(r(s))dW(s) + \int_0^t \int_0^\infty \ln(1 + \bar{\gamma}(r(s), y))\tilde{N}(ds, dy) \right\}, \end{aligned}$$

where

$$(4.6) \quad v(t, x) := \sum_{n=1}^\infty \exp \left\{ -\lambda_n t + \int_0^t \bar{\alpha}(r(s))ds \right\} u_n^0 e_n(x), \quad t \geq 0, \quad x \in \mathcal{O}.$$

Proof. Under (4.3), Eq. (4.2) has a unique global mild solution such that

(4.7)

$$\begin{aligned} X(t) = S(t)u^0 + \int_0^t \bar{\alpha}(r(s))S(t-s)X(s)ds + \int_0^t \bar{\beta}(r(s))S(t-s)X(s)dW(s) \\ + \int_0^t \int_0^\infty \bar{\gamma}(r(s), y)S(t-s)X(s^-)\tilde{N}(ds, dy). \end{aligned}$$

Since

$$S(t)u = \sum_{n=1}^\infty e^{-\lambda_n t} \langle u, e_n \rangle_H e_n \text{ for } u \in H,$$

Eq. (4.7) can be rewritten in the form

$$\begin{aligned} X(t) = \sum_{n=1}^\infty e^{-\lambda_n t} \langle u^0, e_n \rangle_H e_n + \int_0^t \bar{\alpha}(r(s)) \sum_{n=1}^\infty e^{-\lambda_n(t-s)} \langle X(s), e_n \rangle_H e_n ds \\ + \int_0^t \bar{\beta}(r(s)) \sum_{n=1}^\infty e^{-\lambda_n(t-s)} \langle X(s), e_n \rangle_H e_n dW(s) \\ + \int_0^t \int_0^\infty \bar{\gamma}(r(s), y) \sum_{n=1}^\infty e^{-\lambda_n(t-s)} \langle X(s^-), e_n \rangle_H e_n \tilde{N}(ds, dy). \end{aligned}$$

This further yields

$$\begin{aligned} e^{\lambda_n t} \langle X(t), e_n \rangle_H &= \langle u^0, e_n \rangle_H + \int_0^t \bar{\alpha}(r(s)) e^{\lambda_n s} \langle X(s), e_n \rangle_H ds \\ &\quad + \int_0^t \bar{\beta}(r(s)) e^{\lambda_n s} \langle X(s), e_n \rangle_H dW(s) \\ &\quad + \int_0^t \int_0^\infty \bar{\gamma}(r(s), y) e^{\lambda_n s} \langle X(s^-), e_n \rangle_H \tilde{N}(ds, dy). \end{aligned}$$

Then Itô's formula gives

$$\begin{aligned} \langle X(t), e_n \rangle_H &= \langle u^0, e_n \rangle_H \exp \left\{ -\lambda_n t + \int_0^t \left[\bar{\alpha}(r(s)) - \frac{1}{2} \bar{\beta}^2(r(s)) \right] ds \right. \\ &\quad + \int_0^t \int_0^\infty [\ln(1 + \bar{\gamma}(r(s), y)) - \bar{\gamma}(r(s), y)] \lambda(dy) ds \\ &\quad \left. + \int_0^t \bar{\beta}(r(s)) dW(s) + \int_0^t \int_0^\infty \ln(1 + \bar{\gamma}(r(s), y)) \tilde{N}(ds, dy) \right\}. \end{aligned}$$

The desired assertion therefore follows by observing that

$$X(t) = \sum_{n=1}^{\infty} \langle X(t), e_n \rangle_H e_n.$$

□

Theorem 4.3. Under the conditions of Theorem 4.1, Eq. (4.2) has the following properties:

(i) If u^0 is deterministic and $u^0 \neq 0$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) &= -\lambda_{n_0} + \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 \right. \\ &\quad \left. + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right) \quad a.s. \end{aligned}$$

In particular, the solution of Eq. (4.2) with initial condition u^0 will converge exponentially to zero with probability one if and only if

$$\lambda_{n_0} > \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right).$$

(ii) For any initial condition u^0 ,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) &\leq -\lambda_1 + \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 \right. \\ &\quad \left. + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right) \quad a.s. \end{aligned}$$

In particular, the solution of Eq. (4.2) is almost surely exponentially stable if

$$\lambda_1 > \sum_{j=1}^m \pi_j \left(\bar{\alpha}_j - \frac{1}{2} \bar{\beta}_j^2 + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right).$$

Proof. Note that

$$\begin{aligned} \frac{1}{t} \ln(\|X(t)\|_H) &= \frac{1}{t} \ln(\|v(t)\|_H) - \frac{1}{2t} \int_0^t \bar{\beta}^2(r(s)) ds \\ &\quad + \frac{1}{t} \int_0^t \int_0^\infty [\ln(1 + \bar{\gamma}(r(s), y)) - \bar{\gamma}(r(s), y)] \lambda(dy) ds \\ &\quad + \frac{1}{t} M_1(t) + \frac{1}{t} M_2(t), \end{aligned}$$

where

$$M_1(t) := \int_0^t \bar{\beta}(r(s)) dW(s) \text{ and } M_2(t) := \int_0^t \int_0^\infty \ln(1 + \bar{\gamma}(r(s), y)) \tilde{N}(ds, dy).$$

Since

$$\langle M_1 \rangle_t \leq t \max_{1 \leq i \leq m} \beta_i^2 \text{ and } \langle M_2 \rangle_t \leq t \max_{1 \leq i \leq m} \int_0^\infty (\ln(1 + \bar{\gamma}_i(y)))^2 \lambda(dy),$$

together with the strong law of large numbers for local martingales, e.g., Lipster [15], we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} M_1(t) = \lim_{t \rightarrow \infty} \frac{1}{t} M_2(t) = 0 \quad \text{a.s.}$$

Furthermore, by the ergodic property of Markov chains, e.g., [20, Theorem 3.8.1, p126]

$$\begin{aligned} & -\frac{1}{2t} \int_0^t \bar{\beta}^2(r(s)) ds + \frac{1}{t} \int_0^t \int_0^\infty [\ln(1 + \bar{\gamma}(r(s), y)) - \bar{\gamma}(r(s), y)] \lambda(dy) ds \\ & \rightarrow \sum_{i=1}^m \pi_i \left(-\frac{1}{2} \bar{\beta}_i^2 + \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy) \right) \quad \text{a.s.} \end{aligned}$$

whenever $t \rightarrow \infty$. Moreover, it is not difficult to show that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln(\|v(t)\|_H) \begin{cases} = -\lambda_{n_0} + \sum_{j=1}^m \pi_j \bar{\alpha}_j, & \text{if } u^0 \neq 0 \text{ is deterministic,} \\ \leq \lambda_1 + \sum_{j=1}^m \pi_j \bar{\alpha}_j, & \text{for any initial condition } u^0. \end{cases}$$

The proof is therefore complete. \square

Example 4.4. Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with the generator $\Gamma = (q_{ij})_{2 \times 2}$:

$$-q_{11} = q_{12} = \nu > 0, \quad -q_{22} = q_{21} = q > 0.$$

Then the unique stationary probability distribution of the Markov chain $r(t)$ is

$$\pi = (\pi_1, \pi_2) = (q/(\nu + q), \nu/(\nu + q)).$$

For $i = 1, 2$ set

$$\mu_i := \int_0^\infty [\ln(1 + \bar{\gamma}_i(y)) - \bar{\gamma}_i(y)] \lambda(dy).$$

Let $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathbb{R}$ such that

$$(4.8) \quad \bar{\alpha}_1 + \mu_1 > 1, \quad \bar{\alpha}_2 + \mu_2 < 1,$$

and choose q obeying

$$(4.9) \quad 0 < q < \nu(1 - \bar{\alpha}_2 - \mu_2)/(\bar{\alpha}_1 + \mu_1 - 1).$$

Consider the switching-diffusion equation with jumps:

$$(4.10) \quad \begin{cases} dX(t) = (AX(t) + \bar{\alpha}(r(t))X(t))dt \\ \quad + \int_0^\infty \bar{\gamma}(r(t), y)X(t^-)\tilde{N}(dt, dy), t > 0, x \in (0, \pi), \\ X(t, 0) = X(t, \pi) = 0, \quad t > 0, \\ X(0, x) = u^0(x) = \sqrt{2/\pi} \sin x, \quad x \in (0, \pi). \end{cases}$$

The previous stochastic system can be regarded as the combination of two equations

$$(4.11) \quad dX(t) = (AX(t) + \bar{\alpha}_1 X(t))dt + \int_0^\infty \bar{\gamma}_1(y)X(t^-)\tilde{N}(dt, dy), \quad t \geq 0, x \in (0, \pi),$$

and

$$(4.12) \quad dX(t) = (AX(t) + \bar{\alpha}_2 X(t))dt + \int_0^\infty \bar{\gamma}_2(y)X(t^-)\tilde{N}(dt, dy), \quad t \geq 0, x \in (0, \pi)$$

with the same Dirichlet boundary condition and initial condition as Eq. (4.10), switching from one to the other according to the law of the Markov chain. Recalling that $e_n(x) = \sqrt{2/\pi} \sin nx, n = 1, 2, 3, \dots$, are the eigenfunctions of $-A$, with positive and increasing eigenvalues $\lambda_n = n^2$, we hence have $\lambda_1 = 1$. By Theorem 4.1, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) \leq -1 + \pi_1 \bar{\alpha}_1 + \pi_2 \bar{\alpha}_2 + \pi_1 \mu_1 + \pi_2 \mu_2 \quad a.s.$$

This, together with (4.8) and (4.9), yields that Eq. (4.10) is almost surely exponentially stable. On the other hand, note that the initial condition $u^0(x) = \sqrt{2/\pi} \sin x$ is deterministic and $u_1^0 = 1$, which implies $n_0 = 1$. By Theorem 4.3, the solution to Eq.(4.11) has the property

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) = -1 + \bar{\alpha}_1 + \mu_1, \quad a.s.,$$

and the solution to Eq.(4.12) has the property

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln(\|X(t)\|_H) = -1 + \bar{\alpha}_2 + \mu_2, \quad a.s.$$

Then, by (4.8) the solution of stochastic system (4.11) explodes exponentially, and the solution of stochastic system (4.12) converges exponentially to zero. However, we observe that due to the Markovian switching the overall system (4.10) is almost surely exponentially stable.

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